Model Studies of a Strongly Coupled Synchrotron RF System*

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Summary—**The strongly coupled multicavity synchrotron RF system proposed for the 6-bev Cambridge Electron Accelerator operates at 475 Mc and, because of its large size, presents complex problems of assembly and tuning. To achieve a better understanding of the characteristics of the system, a scaled model operating at 9000 Mc was constructed. The work was undertaken in two stages; first a simple single-cavity ring was studied and later a more complicated double-cavity ring. An account is given of the design of cavities and waveguide links of variable electrical length, and instrumentation devised for various measurements is described. This is followed by experimental data which were in good agreement with theoretical predictions. Recommendations arising from the model study for the assembly and tuning of the 475-Mc system are put forward.**

I. Introduction

In ORDER TO gain practical experience on various problems arising from Robinson's[1] proposal of the strongly coupled multicavity synchrotron RF system for the 6-bev Cambridge Electron Accelerator, it was decided to construct an X-band model.

The behavior of a ring structure consisting of 16 strongly coupled double cavities depends on 48 tuning adjustments: the resonant frequencies of each individual half-cavity and the electrical length of the waveguide links joining the double cavities. In order to obtain the desired mode of operation, a systematic tuning method must be established. The desired mode must be identified by cavity voltage measurements, and observations may then be made of the frequency separation between neighboring modes and of the effects on the mode spectrum due to the deliberate detuning of individual components. It was considered feasible to obtain this information with a greatly reduced number of components; for geometrical simplicity a six-cavity structure, intended solely for radio-frequency studies and not suitable for the acceleration of electrons, was chosen. It consisted of a hexagon whose sides were formed by waveguide sections of variable electrical length, with resonant cavities at the vertices. Power was fed to the structure though either a three-port cavity at one of the vertices or a T-junction in one of the waveguide links. The work was undertaken in two stages, first a simple single-cavity ring was studied and later a more complicated double-cavity ring. A diagram of the model assembly is shown in Fig. 1.

* Received by the PGMTT, April 14, 1960; revised manuscript received, June 22, 1960. This work was carried out under AEC Contract AT(30-1)-1909.

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[1] K. W. Robinson, "Radio-frequency system of the Cambridge Electron Accelerator," this issue, p. 593.

II. Design of Components and Measuring Equipment

In order to obtain an axial accelerating field in the synchrotron cavities, the TM_{010} mode is excited and this was also the mode chosen for the individual model cavities. Fig. 2 shows a single cavity. The height of the cavity was chosen to avoid excitation of the unwanted TE_{111} mode. For tuning, a polystyrene rod, mounted on a micrometer head, was inserted through a $\frac{1}{8}$ inch diameter central hole in the top of each cavity. This resulted in a linear tuning range of about $\pm 1\frac{1}{2}$ per cent around the center frequency. The cavities were constructed of triangular blocks of brass. Unloaded Q-

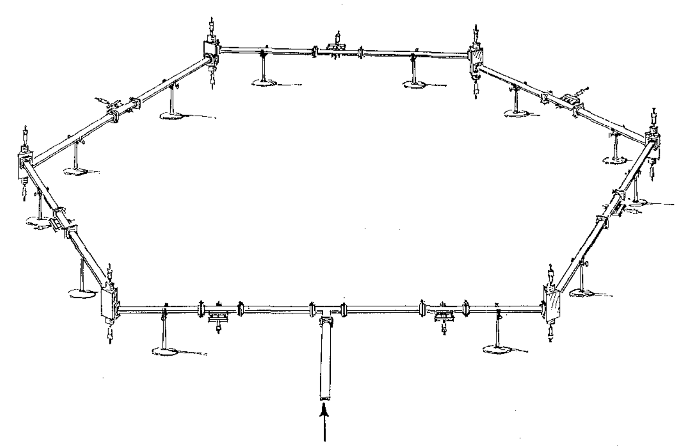

Fig. 1—Model assembly.

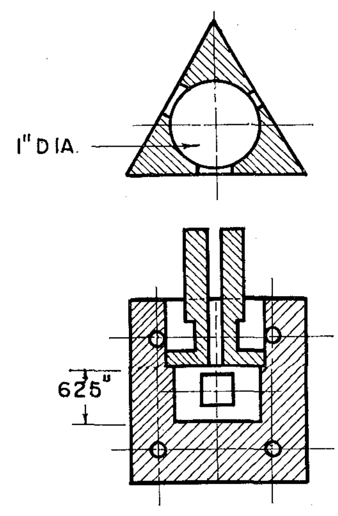

Fig. 2—Single cavity, construction.

values of more than 3000 were obtained with the single cavities and of nearly 4000 with the double cavities as shown in Fig. 3, where the dividing diaphragm between the chambers was made of copper.

An overcoupled double cavity is characterized by two resonant modes which occur at frequencies where the individual cavity voltages are either in phase or in antiphase; *i.e.*, the 0 mode and π mode. For scaling purposes a coupling factor $f_\pi/\Delta f$ was defined where f_π is the π-mode resonant frequency and Δf is the frequency separation between the 0 mode and the π mode. For the coupling factor to be equal to that in the 475-Mc synchrotron cavities, namely 200, Δf equaled 44 Mc in the X-band model. Owing to manufacturing difficulties it was not possible to reproduce this coupling factor consistently with two magnetic coupling slots; however, good reproducibility was obtained by changing from magnetic to electric coupling through a central $\frac{1}{4}$ inch-diameter coupling hole, and this coupling scheme was adopted. The change from magnetic to electric intracavity coupling caused a frequency reversal of the 0 mode and the π mode, the π mode being the higher frequency resonance of the electrically coupled pair.

In order to obtain a satisfactory frequency separation between the desired and adjacent modes of the synchrotron ring, a quantity $1/GZ_0$ equal to 17 was specified by Robinson,[1] where G is the shunt conductance of a cavity represented as a parallel resonant circuit shunted across a waveguide of characteristic impedance Z_0. For a cavity with two matched waveguide terminations, this corresponds to a ratio of unloaded to loaded Q-values equal to 35. In view of the difficulties involved in measing Q-values less than 300 to a reasonable degree of accuracy with the available test equipment, 300 was accepted for the loaded Q-value of both single and double cavities. This corresponds to a value of $1/GZ_0$ approximately equal to 5.

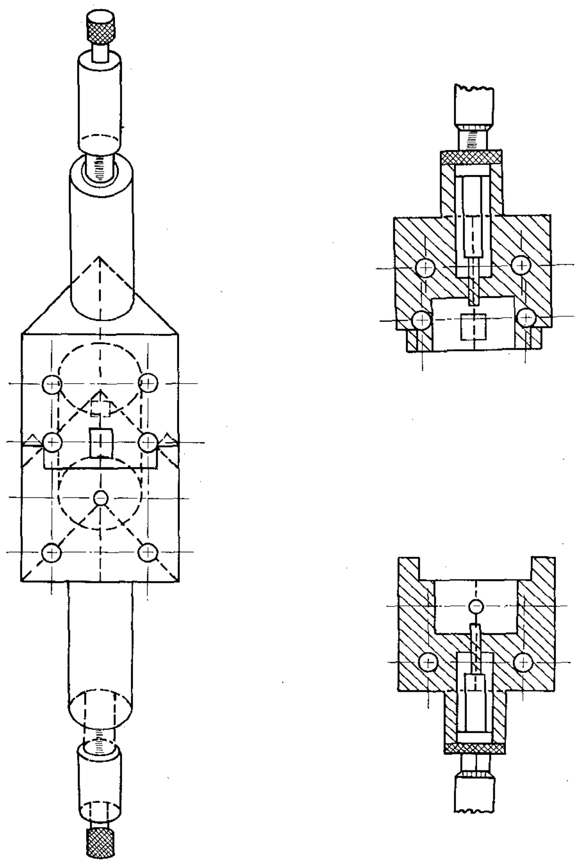

Fig. 3—Double cavity, construction.

The single cavities were tuned by terminating both coupling holes by matched waveguides and observing the resonant frequency as a function of the depth of penetration of the dielectric tuning rods. This tuning method was not sufficiently accurate for the double cavities, since the frequency response of the cavities loaded by matched waveguides was much broader than in the ring structure where the cavities operate under essentially unloaded conditions. A different tuning technique had to be devised in which the cavities were terminated in shorted quarter-wave waveguides so that they were unloaded and their sharpness of tuning was preserved. The details of this procedure are discussed in Section III.

For maximum flexibility the electrical length of the waveguide links joining adjacent cavities was designed for a possible variation exceeding one guide wavelength. For this purpose, a phase shifter was scaled from a Radiation Laboratory design,[2] where a polystyrene vane supported by three correctly spaced metal rods, to achieve cancellation of reflections, could be moved laterally across the interior of the waveguide. At the desired frequency, a vane length of 8 inches gave a phase shift of more than 360°; the voltage standing wave ratio as a function of vane position over the frequency band from 8500 Mc to 9500 Mc did not exceed 1.13. The Q-value of the dielectric loaded section was of the order of 6000 and, within the limits of accuracy of the Q-measuring apparatus, was the same as that of empty copper waveguide.

The frequency response of all components as well as of the assembled resonant ring was measured by means of a reflectometer where the incident power from a frequency-modulated klystron and the power reflected by the component under test were displayed simultaneously on an oscilloscope.

An important feature in studying the behavior of the double cavities and the ring structure was the measurement of RF phase. The scheme adopted is shown in Fig. 4, where two signals whose relative phase is to be measured are introduced into a calibrated slotted line which is terminated by matched attenuators. The superposition of the signals results in a standing wave

[2] G. L. Ragan, "Microwave Transmission Circuits," McGraw-Hill Book Co., Inc., New York N. Y., p. 516; 1948.

pattern, the position of which depends on the relative phase angle. If the position of the standing wave minimum for in-phase signals is taken as the reference phase, then for a phase angle ϕ, the minimum will move to a new position at a distance $(\lambda g/2)$ $(\phi/360)$. The slotted line with its flexible cable connections was calibrated by establishing a minimum position corresponding to a relative phase of 0° with two identical signals obtained from a single cavity with three identical coupling holes as shown. Although the accuracy of the method was greatest with signals of equal amplitude where the position of a standing wave minimum could be measured with great precision, it was possible to make measurements with signals of arbitrary magnitude.

III. Tuning Procedure of the Ring Structure

In order to establish the correct electrical length of the waveguide links, a precise determination of the cavity planes is required. Near the resonant frequency, a two-port cavity may be represented by a four-terminal network as shown in Fig. 5, where the effect of the coupling holes is represented by lumped reactances X. If the cavity is excited through one port and connected to an adjustable short-circuited waveguide through the other port, it is possible to determine the equivalent electrical length of the combination of lumped reactance X and shorted line by monitoring the cavity voltage as the line length is varied. For constant input power the cavity voltage will change from a maximum to a minimum as the equivalent line length is changed. In practice, it was difficult to observe the maximum where the cavity voltage was not sentitive to changes of line length, but deep minima were obtained. If X is small, the effective cavity plane a-b may then be taken to be one-half guide wavelength from the face of the shorting plunger.

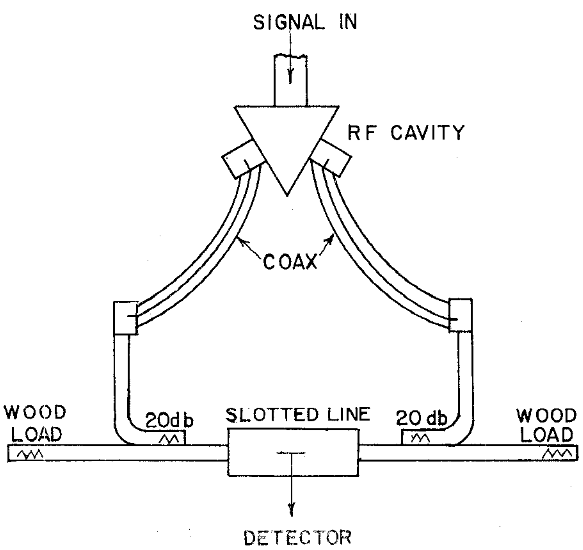

Fig. 4—Calibration of slotted-line phasemeter.

It was first required to ascertain the distance between the effective plane of the shorting plunger and its flange from a slotted-line measurement. The cavity whose effective plane was to be measured was then connected as shown in Fig. 6. The minimum cavity voltage corresponded to a position of the shorting plunger approximately one-half guide wavelength from the inner cavity wall. This result was confirmed with a slotted-line measurement. The data obtained were in good agreement for the two sets of six cavities.

Regarding the tuning of the double cavities, it can be shown that in the desired mode of operation of the ring structure, a voltage minimum exists in the ring at a distance of approximately one-quarter of a guide wavelength from the plane were a half-cavity may be represented by an equivalent lumped parallel resonant circuit. If the circuit is considered to be lossless, the standing wave pattern in the ring remains theoretically undisturbed by inserting a metal wall into the waveguide at this point. If the position of the effective cavity plane is known, this condition may be simulated by means of two adjustable shorting plungers in waveguide arms connected to the coupling holes of the individual half-cavities. In the experiment, the plungers were adjusted to one-quarter guide wavelength from the effective cavity planes; each half-cavity was provided with two small coupling holes such that power could be fed through one and the cavity voltage could be monitored through the other. The tuning procedure consisted of feeding power at the desired frequency into one of the half-cavities, displaying both half-cavity voltages simultaneously and recording the position of the tuning rods for equal voltage maxima.

The electrical length of the waveguide links joining

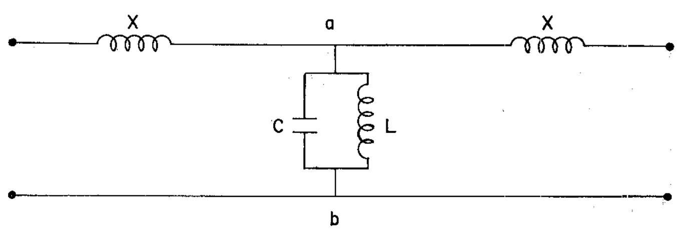

Fig. 5—Equivalent circuit of two-port cavity.

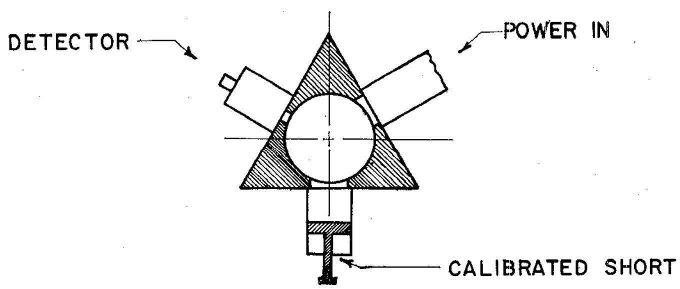

Fig. 6—Determination of cavity plane.

adjacent cavities was chosen to be about $18\frac{1}{2}$ guide wavelengths, approximately the same as the electrical distance between cavities in the synchrotron ring. By means of the phase shifters described previously, this quantity could be varied over a range of over 360°. In order to obtain the optimum frequency spacing between the desired fundamental mode and the resonant frequencies of the other, nonaccelerating modes, an electrical length of the waveguide links was specified differing from an integral number of half wavelengths by a small angle δ. The method used for adjusting the links to the required electrical length was as follows: A typical link consisted of two empty sections, 14.42 inches long, mounted on either side of a phase shifter, 8.5 inches long, giving a total length of 37.34 inches, corresponding to $18\frac{1}{2}$ guide wavelengths in empty guide at 8700 Mc. The link was terminated at one end with a thin copper diaphragm containing a small coupling hole and at the other with an adjustable short circuit. The position of the shorting plunger, when it produced an electrical short circuit at its flange face, was designated the "0" position. For a positive angle δ and a distance d of the effective cavity plane from the face of the cavity, the plunger was set at "0"$+2d+\delta\lambda_g/360$ inches. The resonant frequency of the structure fed as a resonator through the small coupling hole was adjusted to the ring operating frequency.

For a phase shift range over 360°, resonant positions were found corresponding to phase shifts from flange to flange of approximately 0° or 180°. The link was then connected between two cavities, one of which was terminated by a matched load and the other was driven by the generator. The desired position of the phase shifter was identified by a phase measurement of the cavity voltages. This procedure was adequate for the single-cavity ring. In the double-cavity case where cavity tuning was more sensitive and complicated, the following scheme was adopted: Each cavity was tuned individually by means of quarter-wave plungers, a link was then tuned to the desired length and connected between two cavities. The frequency at which the correct phase relation between cavities obtained was now different from the resonant frequency of individual cavities. All other links, when terminated by two tuned cavities and adjusted to this new frequency, were then of the correct electrical length. The two frequencies were identical only when δ was equal to zero.

Regarding the tuning of the input link, a T-junction may be represented at a fixed frequency by the equivalent circuit shown in Fig. 7. If phase shifters are included in arms I and II of the T and these are terminated by a diaphragm containing a small coupling hole and an adjustable short circuit, respectively, at a given frequency the phase shifters can be adjusted such that maximum power may be coupled into arm I while essentially no power is coupled into arm III. Under this condition the link is an integral number of half-wavelengths long and, of course, power fed through arm III then divides symmetrically into arms I and II, if these are terminated identically. In order to simulate the cavity planes, arm I was lengthened by a shim of the appropriate thickness and arm II was suitably compensated by adjusting the micrometer setting of the adjustable short. Two cases were of interest, 1) where arms I and II were equal and 2) where their electrical lengths differed by one-half guide wavelength, corresponding to 0° and 180° excitation, respectively. Resonance of the link in the desired mode pattern was then indicated by a deep null on the detector output meter in arm III and by minimum reflection from arm I.

IV. Experimental Results

The characteristics of the ring structure can be computed by means of a matrix formulation (see Appendix).

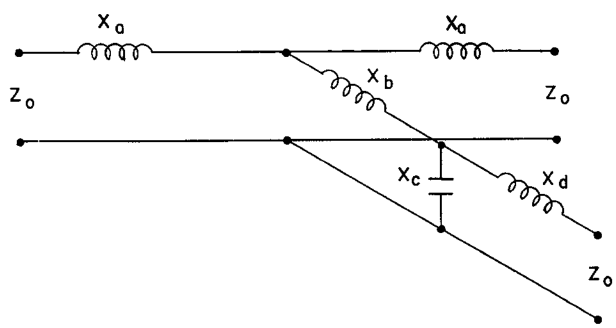

Fig. 7—Equivalent circuit of T junction.

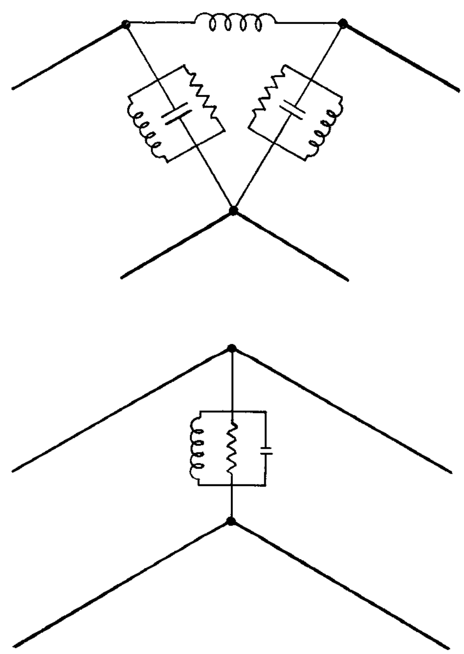

Fig. 8—Equivalent circuit of double cavity (upper diagram), and single cavity representation at either resonance (lower diagram).

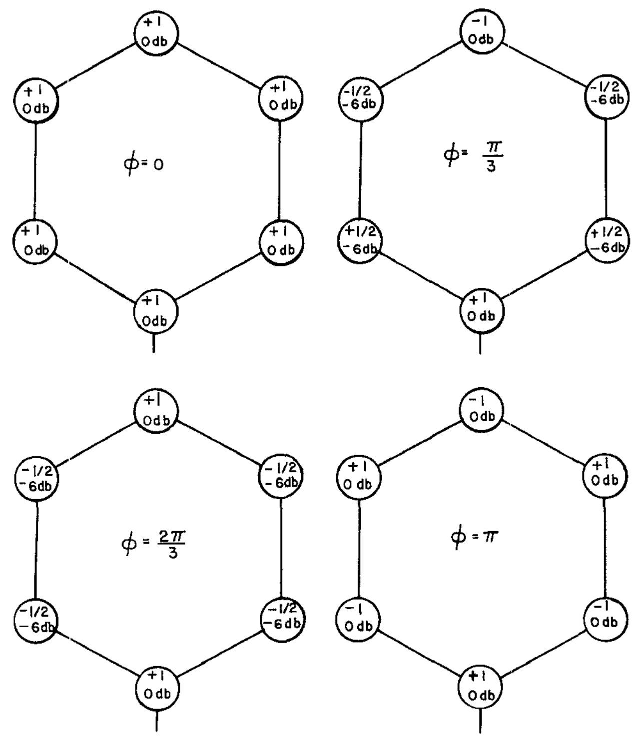

Fig. 9—Amplitude distribution of modal cavity voltages (cavity-fed single-cavity ring).

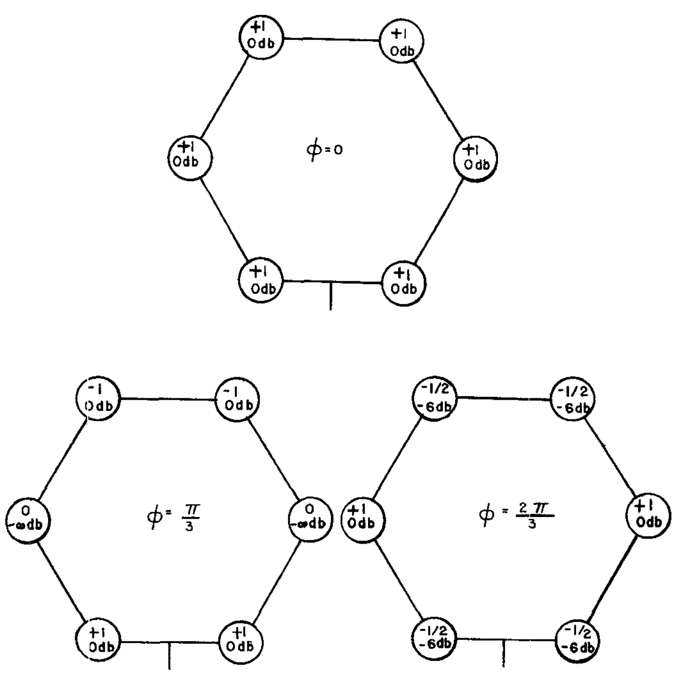

Fig 10—Amplitude distribution of modal cavity voltages (T-fed single-cavity ring).

Since the cavities are strongly coupled, a double cavity at either of its resonances may be represented by a single cavity (Fig. 8). This simplifies the analysis considerably and enables one to obtain the characteristics of the system by means of a hand calculation. For this reason, the first set of experiments was carried out with a system consisting of single cavities. In order to reduce the number of variables, the ring was initially fed through a three-port cavity with identical coupling holes.

At resonance, the phase shift, ϕ, between adjacent cavities is limited to values of ϕ equal to 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$ radians. Pairs of degenerate modes occur at some frequencies and their amplitudes add. The amplitude distribution of the modal voltages depends on the location where power is fed into the system. For the cavity-fed ring, the theoretical amplitude distribution of the cavity voltages is shown in Fig. 9. For convenience, the mode with $\phi=0$, where all cavities oscillate in phase, was chosen as the operating mode. Similarly, for the T-fed ring the cavity voltages vary according to Fig. 10, where it is seen from symmetry considerations that the mode with $\phi=\pi$ cannot be excited.

A typical mode-pattern obtained in terms of power reflected from the input of the ring as a function of frequency is shown in Fig. 11(a); two sets of modes appear on either side of the operating mode ($\phi=0$) resonance. For comparison, the theoretical modal frequencies are indicated in terms of frequency separation from the desired $\phi=0$ mode in Fig. 11(b).

A mode corresponding to the resonant frequency of the waveguide link, the "waveguide-mode" (WG), was calculated to be separated from the operating mode by 6 Mc. This was confirmed experimentally; this mode was unaffected by a detuning of the cavities but was greatly influenced by the individual phase shifters.

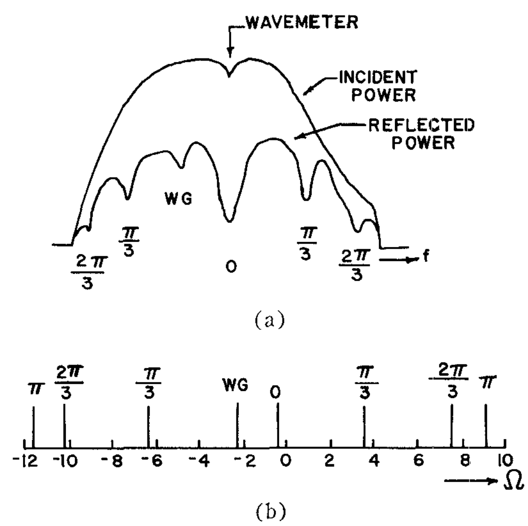

Fig. 11—(a) Experimental resonance pattern of single-cavity ring. (b) Theoretical modal resonances of single-cavity ring. Ω=frequency separation in cavity bandwidths from the operating ($\phi=0$) mode.

The calculated and measured modal resonant frequencies are shown in Table I; the amplitude distribution of cavity voltages in decibels for both cavity-fed and T-fed rings is given in Table II, the maximum deviation from the mean being 0.73 db for the cavity-fed ring and 0.75 db for the T-fed ring in the desired $\phi=0$ resonance. In view of the fact that the position of the dielectric vane in the phase shifters was not the same in the six links, resulting in a variation of the dispersion from link to link, the discrepancy between experimental and theoretical data regarding the other modes was surprisingly small.

A change of the resonant frequency Δf of a single cavity was shown by Robinson[1] to produce a corresponding change $\Delta f/N$ in the resonant frequency of a ring consisting of N cavities. This was confirmed experimentally for the operating mode of the T-fed ring, where cavity no. 1 was detuned in three steps to a maximum of 44 Mc. The change of the operating mode frequency under this condition was 7 Mc. The amplitude distribution of the cavity voltages for the three changes of no. 1 cavity resonant frequency is shown in Table III.

Since the synchrotron ring is required to operate in the π-π mode (180° phase difference between successive half-cavities), this was the mode most thoroughly investigated with the double-cavity ring. As in the single-cavity experiments, power was fed into the ring through either a cavity or a T-junction. The results of amplitude-distribution measurements and detuning experiments are summarized in Tables IV and V, from which it is seen that the behavior of the ring was essentially the same as that of the single-cavity structure. Apart from the desired fundamental mode, it was not possible to identify the resonant modes by their amplitude distributions. This was probably due to mechanical imperfections of the double cavities, particularly to ran-

TABLE I

FREQUENCIES OF MODAL RESONANCES

Modal resonance ϕ	Resonant frequency (Mc)	
	Calculated	Measured
0	8698.8	8700
$\pi/3$	8710.4, 8681.7	8711, 8684
$2\pi/3$	8722.0, 8670.4	8722, 8675
π	8726.4, 8666.4	8726, 8670

TABLE II

AMPLITUDE DISTRIBUTION OF CAVITY VOLTAGES

	Modal resonances				
	$\phi=0$		$\phi=\pi/3$		$\phi=2\pi/3$
Cavity No.	Cavity-fed*	Tee-fed	Cavity-fed*	Tee-fed	Tee-fed
1	—	0	—	0. (0)	−6.0 (−6)
2	0	−2.0	−4.5 (−6)	−10.3 (−∞)	0 (0)
3	−1.45	−1.0	−9.0 (−6)	− 1.2 (0)	−4.0 (−6)
4	−0.60	−1.5	0 (0)	− 0.6 (0)	−5.6 (−6)
5	−0.95	−1.0	−4.0 (−6)	− 9.8 (−∞)	−1.0 (0)
6	−0.60	−1.0	−5.0 (−6)	− 2.8 (0)	−3.5 (−6)
	Cavity voltages in decibels—reference level 0 decibels				

* No provision for voltage monitoring of feed-cavity. Bracketed figures are theoretical values.

TABLE III

AMPLITUDE DISTRIBUTION OF OPERATING-MODE CAVITY VOLTAGES (RESONANT FREQUENCY OF CAVITY NO. 1 DECREASING)

	Resonant frequency of cavity no. 1 decreased by			
Cavity No.	0 Mc	14.6 Mc	29.2 Mc	44 Mc
1	0	0.1	−1.8	−3.9
2	−2.2	−2.2	−3.7	−4.8
3	−1.2	−1.4	−2.8	−3.0
4	−1.6	−2.0	−2.8	−3.0
5	−1.2	−1.5	−1.4	−3.0
6	−0.6	−0.4	−1.5	−2.7
	Cavity voltages in decibels—reference level 0 decibel			

TABLE IV

AMPLITUDE DISTRIBUTION OF π-π MODE CAVITY VOLTAGES, INITIAL CAVITY TUNING 8852 MC

Double Cavity No.	Upper	Lower	Resonant frequency of cavity no. 5: Decreased by 17 Mc, Upper	Decreased by 17 Mc, Lower	Increased by 6.5 Mc, Upper	Increased by 6.5 Mc, Lower
1*	—	—	—	—	—	—
2	−2.0	−1 2	−3.3	−4 2	−2.3	−1.2
3	−1.6	−1.8	−2.5	−4.0	−1.4	−1.8
4	−1.7	−2.4	0	−2.0	0	−2.0
5	−2 6	−2.2	−0.2	0	−1.4	−0.8
6	−1.9	−2.8	−3.3	−0.2	−1.4	−0.4
Modal resonant frequency	8851 Mc		decreased by 2 Mc		increased by approx. ½ Mc	
Maximum deviation from mean, db	0.8		2.2		1.15	

* Cavity-fed ring, no provision for voltage monitoring of feed cavity. Surplus electrical length of waveguide link $\delta=0$. Cavity voltages in decibels, reference level 0 decibels.

TABLE V

AMPLITUDE DISTRIBUTION OF π-π MODE CAVITY VOLTAGES FOR TWO VALUES OF SURPLUS ELECTRICAL LENGTH δ OF WAVEGUIDE LINK

	Surplus electrical length of waveguide link			
	$\delta=-15°$		$\delta=+15°$	
Double Cavity No.	Upper	Lower	Upper	Lower
1*	—	—	—	—
2	−0.2	0.5	−1.0	−2.4
3	0	0	−0.8	−1.6
4	0	−0.4	0	−1.6
5	−0.8	0	−1.6	−1.7
6	0.5	0	−0.6	−1.7
Maximum deviation from mean, db	0.65		1.2	

* Cavity-fed ring, no provision for voltage monitoring of feed cavity. Cavity voltages in decibels, reference level 0 decibels.

dom distortions of the copper diaphragm forming the separating wall. It was possible, however, to compare experimental results with theoretical predictions of the mode spectrum by means of phase measurements. The computation of the modal resonances in this case was carried out on a digital computer. Both experimental and theoretical data are shown in Fig. 12. The results are plotted in terms of frequency separation from the desired $\phi = 180°$ mode as a function of the electrical length of the waveguide links around an odd integral number of half wavelengths. As in the single cavity case, the waveguide mode was identified by its sensitivity to changes in phase shifter setting. It was not possible to detune any double cavity by more than 5 bandwidths without seriously disturbing the mode pattern.

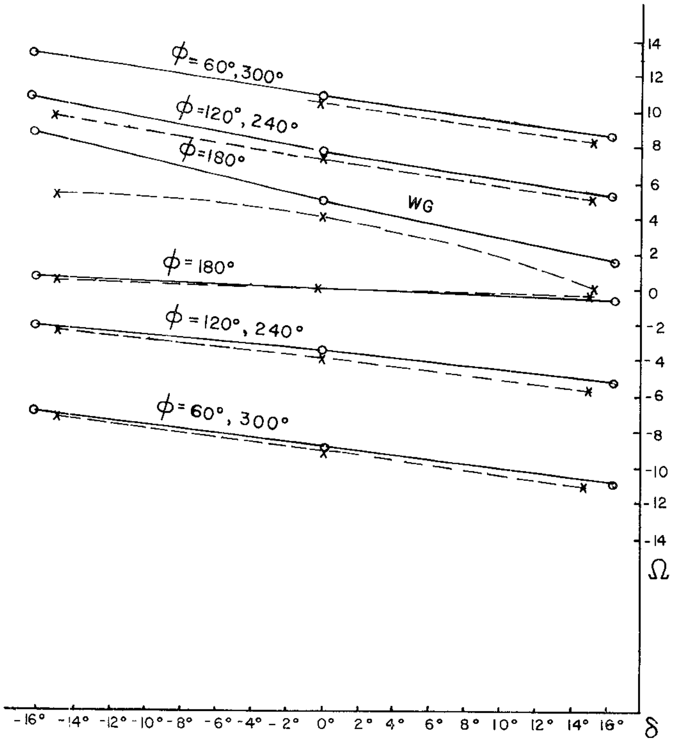

Fig. 12—Experimental and theoretical modal resonances of double-cavity ring. Ω = frequency separation in cavity bandwidths from the operating ($\phi = 180°$) mode; δ = excess electrical length of waveguide links over nearest integral number of half-wavelengths.

V. Conclusions

The experiments described essentially confirm the theoretical predictions regarding the ring structure and demonstrate its feasibility for accelerating purposes.

However, considering the mechanical complexity of the synchrotron ring whose average radius is 118 feet, it is desirable that individual components can be tuned without removal from the ring. From the model investigation of the double-cavity tuning which required quarter wavelength shorts to simulate the unloaded conditions of the cavities, a method has been devised to accomplish this.

The scheme involves shorting the waveguide link joining two cavities, which is approximately 50 feet long, at appropriate positions by means of resonant posts and it depends ultimately on the reflection coefficient of the posts. An experimental post has been built for operation in 18-inch × 9-inch waveguide at 475 Mc, and a VSWR of better than 40 db was obtained. For tuning of the cavities or of the waveguide links, the posts will be placed at an odd-integral number of quarter wavelengths from the effective cavity planes; the cavities and waveguide links will then be adjusted as required with a low-power signal. Once the components are tuned, the posts are withdrawn from the guide.

It is proposed to equip all waveguide sections with such posts, mounted on sliding carriages in order to adjust their position. The tuning procedure will otherwise be identical with that outlined in the paper.

Appendix

One of the most important sets of parameters characterizing the strongly coupled multicavity RF system is its group of resonant frequencies in the desired frequency range. These frequencies are intimately related to the circuit parameters in theory, and they are readily available as experimental data.

Since the RF system under investigation is a high-Q circuit, the resonant frequencies will not be affected greatly by circuit losses. Consequently, the frequency spectrum may be calculated from an eigenvalue formulation.[3]

Consider the resonant ring circuit from which the losses and the input junction have been deleted. The circuit has no terminals. Suppose, now, that the ring is broken at some point in the circuit so that a two-terminal-pair network is produced. This circuit is described by a matrix of general circuit parameters[4] relating the voltage and current at the terminals on one side of the network to the voltage and current at the terminals on the other side. The matrix elements are functions of the frequency and the resonances occur at frequencies where the voltage and current at one port are reproduced at the other port.

If the ring consists of n identical cells, each of which is itself symmetrical, then the matrix for a cell may be written,

$$m = \begin{pmatrix} \cos\phi & b \\ c & \cos\phi \end{pmatrix},$$

and the matrix of the opened ring,

$$m^N = \begin{pmatrix} \cos N\phi & b \sin N\phi / \sin\phi \\ c \sin N\phi / \sin\phi & \cos N\phi \end{pmatrix} = M;$$

the elements b and c are imaginary because of conservation of energy.

[3] G. B. Collins, "Microwave Magnetrons," McGraw-Hill Book Co., Inc., New York, N. Y., p. 123; 1948.

[4] "Reference Data for Radio Engineers," International Telephone and Telegraph Corp., New York, N. Y., 4th ed., p. 143; 1956.

The general eigenvalue problem may be expressed as $M\psi = \lambda\psi$, where ψ is the voltage-current eigenvector and λ is the eigenvalue. For resonance, $\lambda = 1$ and hence $\cos N\phi = 1$, from which $\phi = 2\pi(n/N)$ $(n = 0, 1, 2, \cdots, N-1)$.

The elementary cell for the single-cavity ring may be chosen in two ways, as shown in Fig. 13.

With either choice,

$$\cos\beta - GZ_0\Omega \sin\beta = \cos\phi$$

yields the resonant frequencies. A graphical solution using the parameters of the model:

$$\omega = 2\pi \times 8.7 \times 10^9 \text{ radians/sec},$$
$$\Omega = Q\Delta\omega/\omega,$$
$$1/GZ_0 = 5,$$
$$\beta(\omega_0) = 36\pi + 0.2 \text{ radians},$$

yields the spectrum in the neighborhood of the fundamental mode as shown in Fig. 11(b).

The elementary cell network for the double-cavity ring is shown in Fig. 14.

The equation for the resonant frequencies in this case is:

$$\cos\beta\left(1 + \frac{GZ_0}{SZ_0}\Omega\right) - \sin\beta\left(\frac{G^2Z_0^2}{2SZ_0}\Omega + GZ_0\Omega - \frac{1}{2SZ_0}\right) = \cos\phi,$$

where

$$\phi = 2\pi n/N \ (n = 0, 1, 2, \cdots, N-1),$$
$$\Omega = Q\Delta\omega/\omega,$$
$$\beta(\Omega) = \beta_0(1 + \Omega\Delta/Q),$$
$$\Delta = 1 + (\lambda_{g0}/\lambda_c)^2 = (\lambda_{g0}/\lambda)^2,$$

λ_{g0} = guide wavelength at ω_0,

λ_c = cutoff wavelength,

$S = 1/2X$, where X is the coupling reactance.

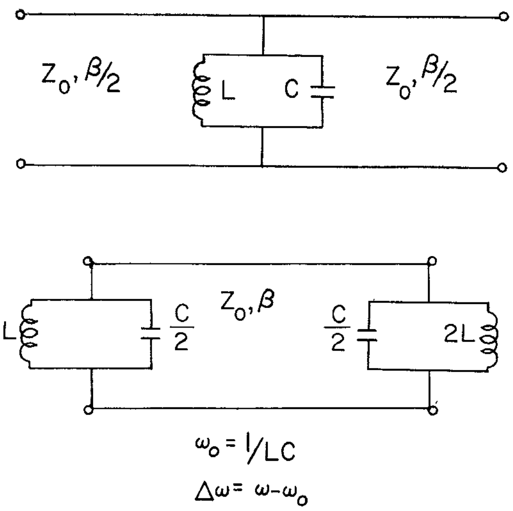

Fig. 13—Elementary cell representations of single-cavity ring.

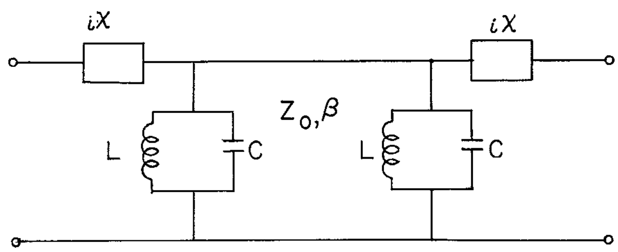

Fig. 14—Elementary cell representation of double-cavity ring.

This equation was solved on the Harvard Univac Electronic Computer and the results are plotted in Fig. 12 for the parameters of the model:

$$1/GZ_0 = 4.87,$$
$$SZ_0 = 2.5,$$
$$Q = 4033,$$
$$\beta_0 = 37\pi + \delta,$$
$$\Delta/Q = 0.000563.$$